3.1 $X(n)$ ergodic consists of independent stochastic variables:

$E[X(n)] = m_x \quad E[X^2(n)] = \sigma_x^2 + m_x^2 \quad \forall n$

WSS + independence $\quad E[X(n_1)X(n_2)] = m_x^2 \quad \forall n_1, n_2 \quad n_1 \neq n_2$

$$r_x(k) = \underbrace{\begin{cases} \sigma_x^2 + m_x^2 & k=0 \\ m_x^2 & k \neq 0 \end{cases}}_{\text{Stationary!}} \qquad Y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(n-k)$$

(a) Variance σ_y^2 of $Y(n)$:

$$\bullet\ E[(Y-m_y)^2] = E[Y^2(n)] - m_y^2 \rightarrow \left\{ m_y = E[Y(n)] = E\left[\frac{1}{N}\sum_{k=0}^{N-1} X(n-k)\right] = \{\text{linearity}\} = \frac{1}{N} E\left[\sum_{k=0}^{N-1} X(n-k)\right] = \{\text{WSS}\} = \frac{1}{N} N \cdot m_x = \underline{m_x} \right\}$$

$Y(n)$ is an unbiased estimator!

$$E[Y^2(n)] = E\left[\frac{1}{N}\sum_{k=0}^{N-1} X(n-k)\, \frac{1}{N}\sum_{j=0}^{N-1} X(n-j)\right] \underset{\text{linearity}}{=} \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} E[X(n-k)X(n-j)] =$$

$$= \left\{\begin{array}{l}\text{to use independence we} \\ \text{need to separate the} \\ \text{cases with } k=j \text{ and} \\ k \neq j\end{array}\right\} = \frac{1}{N^2}\underbrace{\sum_{k=0}^{N-1}}_{N\text{ cases}}\ \cdot \underbrace{\sum_{j=k}}_{1\text{ case}} E[X(n-k)^2] +$$

$$+ \frac{1}{N^2}\underbrace{\sum_{k=0}^{N-1}}_{N\text{ cases}}\ \cdot \underbrace{\sum_{j\neq k}}_{(N-1)\text{ cases}} E[X(n-k)X(n-j)] = \frac{1}{N^2}\underbrace{\sum_{k=0}^{N-1}}_{N\text{ cases}} \underbrace{r_x(0)}_{\sigma_x^2+m_x^2} + \frac{1}{N^2}\underbrace{\sum_{k=0}^{N-1}}_{N}\underbrace{\sum_{j\neq k}}_{N-1} \underbrace{r_x(k-j)}_{m_x^2} =$$

$$= \frac{1}{N^2} N(\sigma_x^2 + m_x^2) + \frac{1}{N^2} N(N-1) m_x^2 =$$

$$= \frac{1}{N}(\sigma_x^2 + m_x^2) + \frac{(N-1)}{N} m_x^2 = \frac{1}{N}\sigma_x^2 + m_x^2$$

$$\sigma_y^2 = \frac{1}{N}\sigma_x^2 + m_x^2 - m_x^2 = \frac{1}{N}\sigma_x^2$$ Consistent estimator!

(i.e. $\sigma_y^2 \to 0$ when $N \to \infty$)

(b) σ_y^2 of $Y(n)$ by considering $Y(n)$ as an output of a linear system:

$$h(m) = \frac{1}{N}\sum_{k=0}^{N-1} \delta(m-k) \qquad r_y(\ell) = \underbrace{r_x(\ell) * h(\ell)} * h(-\ell)$$

$$\sigma_y^2 = r_y(0) - m_y^2 \qquad r_x(\ell) * h(\ell) = \frac{1}{N}\sum_{m=0}^{N-1} r_x(\ell-k)$$

$$h(-\ell) = \frac{1}{N}\sum_{k=0}^{N-1} \delta(-\ell-k) = \frac{1}{N}\sum_{k=0}^{N-1} \delta(\ell+k)$$

$$r_x(\ell) * h(\ell) * h(-\ell) = \frac{1}{N}\sum_{k=0}^{N-1} r_x(\ell-k) * \frac{1}{N}\sum_{j=0}^{N-1} \delta(\ell+j) =$$

$$= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} r_x(\ell+j-k) = r_y(\ell)$$

since $\sigma_y^2 = r_y(0) - m_y^2 =$

$$r_y(0) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} r_x(j-k) \;//$$ same as in (a)!

3.2 $X(n)$ sequence of independent stochastic variables:

- Ergodic with ACF: $r_x(k) = \sigma_x^2\delta(k) \Rightarrow$ uncorrelated with all others! + $\emptyset$ mean
  white!

σ_x^2 is estimated by the averaging $Z(n) = \frac{1}{N}\sum_{k=n-N+1}^{n} X^2(k)$

(a) Estimate $Z(n)$ of σ_x^2 unbiased?

$$E[Z(n)] = \frac{1}{N}\sum_{k=n-N+1}^{n} E[X^2(k)] = \frac{1}{N}\sum_{k=n-N+1}^{N} r_x(0) = \frac{N}{N} r_x(0) = \sigma_x^2$$

unbiased!

(b) σ_z^2 of $Z(n)$ as a function of the 4th order moment of $X(n)$:

$E[X^4(n)] = N_4$

$$E[Z^2(n)] = E\left[\frac{1}{N}\sum_{k=n-N+1}^{n} X^2(k)\, \frac{1}{N}\sum_{j=n-N+1}^{n} X^2(j)\right] = \{\text{linearity}\} =$$

$$= \frac{1}{N^2}\sum_{k=n-N+1}^{n}\sum_{j=n-N+1}^{n} E[X^2(k)X^2(j)] = \frac{1}{N^2}\sum_{k=n-N+1}^{n} \underbrace{E[X^4(k)]}_{j=k} +$$

$$+ \frac{1}{N^2}\sum_{k=n-N+1}^{n}\sum_{j\neq k} \underbrace{E[X^2(k)]E[X^2(j)]}_{\text{independence}} = \frac{\cancel{N}}{N^{\cancel{2}}} N_4 + \frac{\cancel{N}(N-1)}{N^{\cancel{2}}}\sigma_x^4$$

$$= \frac{N_4}{N} + \frac{N-1}{N}\sigma_x^4$$

$$\sigma_z^2 = E[Z(n)^2] - \sigma_x^4 = \frac{N_4}{N} + \frac{N-1}{N}\sigma_x^4 - \frac{N}{N}\sigma_x^4 =$$

$$= \frac{N_4}{N} - \frac{\sigma_x^4}{N} = \frac{1}{N}\left(N_4 - \sigma_x^4\right)$$

From charts we know that for a $\emptyset$ mean Gaussian random variable
$E[X^4] = 3(E[X^2])^2$

(c) $\sigma_z^2 = \frac{1}{N}(\mu_4 - \sigma_x^4) = \frac{1}{N}(3\sigma_x^4 - \sigma_x^4) = \frac{2}{N}\sigma_x^4$

$\sigma_z^2 \leq 0.01\,\sigma_x^4 \Rightarrow \frac{2}{N}\sigma_x^4 \leq 0.01\,\sigma_x^4$

$\frac{2}{0.01} \leq N \qquad \boxed{200 \leq N}$

EXERCISE 1 EXAM 29/10/2013

1. $X_1, X_2, \dots$ are independent random variables
X_i is Gaussian $m > 0$ and variance σ^2
$Y = \sum_{i=1}^{\infty} (a)^i X_i, \quad |a| < 1$

(a) Find the probability $P(Y > 0)$ as a function of m, σ^2 and a.

Y is a linear combination of independent Gaussian random variables $\Rightarrow$ Y is also Gaussian $\Rightarrow$ can be completely characterized from its mean and variance.

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-m_y)^2}{2\sigma_y^2}}$$

$$m_y = E[Y] = E\left[\sum_{i=1}^{\infty}(a)^i X_i\right] = \sum_{i=1}^{\infty}(a)^i E[X_i] = \sum_{i=1}^{\infty}(a)^i m =$$

$$= m\sum_{i=1}^{\infty}(a)^i \underset{|a|<1}{=} m\left(\frac{a}{1-a}\right) = \frac{ma}{1-a}$$

→ Variance:

$$\sum_{i=1}^{\infty}(a)^i x_i = \sum_{i=1}^{\infty} z_i \qquad \sigma_y^2 = \sum_{i=1}^{\infty}\sigma_{z_i}^2 = \sum_{i=1}^{\infty}(a)^{2i}\sigma_x^2 \underset{|a^2|<1}{=} \sigma_x^2\left(\frac{ma^2}{1-a^2}\right)$$

$$z_i = (a)^i x_i \longrightarrow \sigma_{z_i}^2 = (a)^{2i}\sigma_x^2$$

$$P(Y>0) = \int_0^\infty f_y(y)\,dy = \int_0^\infty \frac{1}{\sqrt{2\pi \cdot \frac{\sigma^2 a^2}{1-a^2}}} e^{-\frac{\left(y-\frac{ma}{1-a}\right)^2}{2\cdot\frac{\sigma^2 a^2}{1-a^2}}} dy$$

change of variables

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\,dt$$

$$b = \frac{y - \frac{ma}{1-a}}{\sigma \frac{a}{\sqrt{1-a^2}}} \qquad db = \frac{dy}{\sigma \frac{a}{\sqrt{1-a^2}}}$$

$$P(Y>0) = \int_{\frac{-\frac{ma}{1-a}}{\sigma\frac{a}{\sqrt{1-a^2}}}}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\frac{\sigma a}{\sqrt{1-a^2}}}} e^{-\frac{b^2}{2}} \cancel{\frac{\sigma a}{\sqrt{1-a^2}}} \cdot db = \boxed{Q\left(\frac{-\frac{ma}{1-a}}{\sigma\frac{a}{\sqrt{1-a^2}}}\right)}$$

$$= Q\left(\frac{-m\sqrt{1-a^2}}{\sigma(1-a)}\right) = Q\left(\frac{-m\sqrt{(1+a)\cancel{(1-a)}}}{\sigma\sqrt{(1-a)\cancel{(1-a)}}}\right) = \boxed{Q\left(-\frac{m}{\sigma}\sqrt{\frac{(1+a)}{(1-a)}}\right)}$$

(b) Find a such that $P(Y>0) = P(X_1>c)$

$$P(X_1>c) = \int_c^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}} = \left\{ b = \frac{(x-m)}{\sigma} \right\} =$$

$$= \int_{\frac{c-m}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b^2} db = Q\left(\frac{c-m}{\sigma}\right)$$

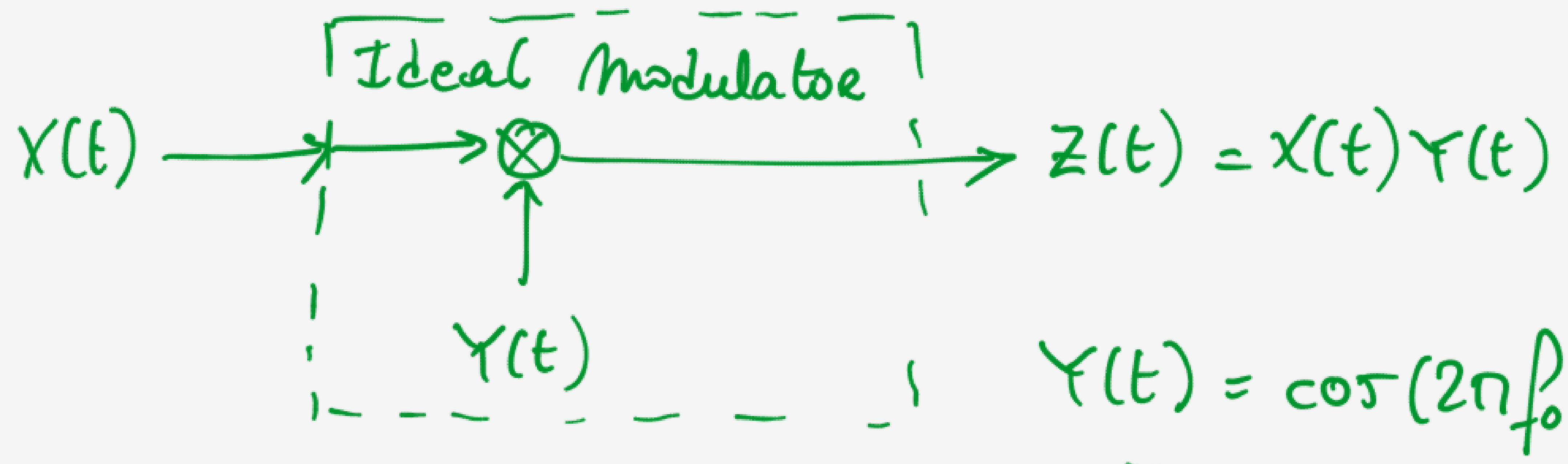
$$-\frac{m}{\sigma}\sqrt{\frac{(1+a)}{(1-a)}} = \frac{c-m}{\sigma} \Rightarrow \sqrt{\frac{(1+a)}{(1-a)}} = \frac{m-c}{m}$$

$$\frac{1+a}{1-a} = (1-c/m)^2 \qquad 1+a = (1-a)(1-c/m)^2$$

$$1+a = (1-c/m)^2 - a(1-c/m)^2 \qquad a(1+(1-c/m)^2) = (1-c/m)^2 - 1$$

$$\boxed{a = \frac{(1-c/m)^2 - 1}{(1-c/m)^2 + 1}}$$

EXERCISE 2 : EXAM 08/01/2014

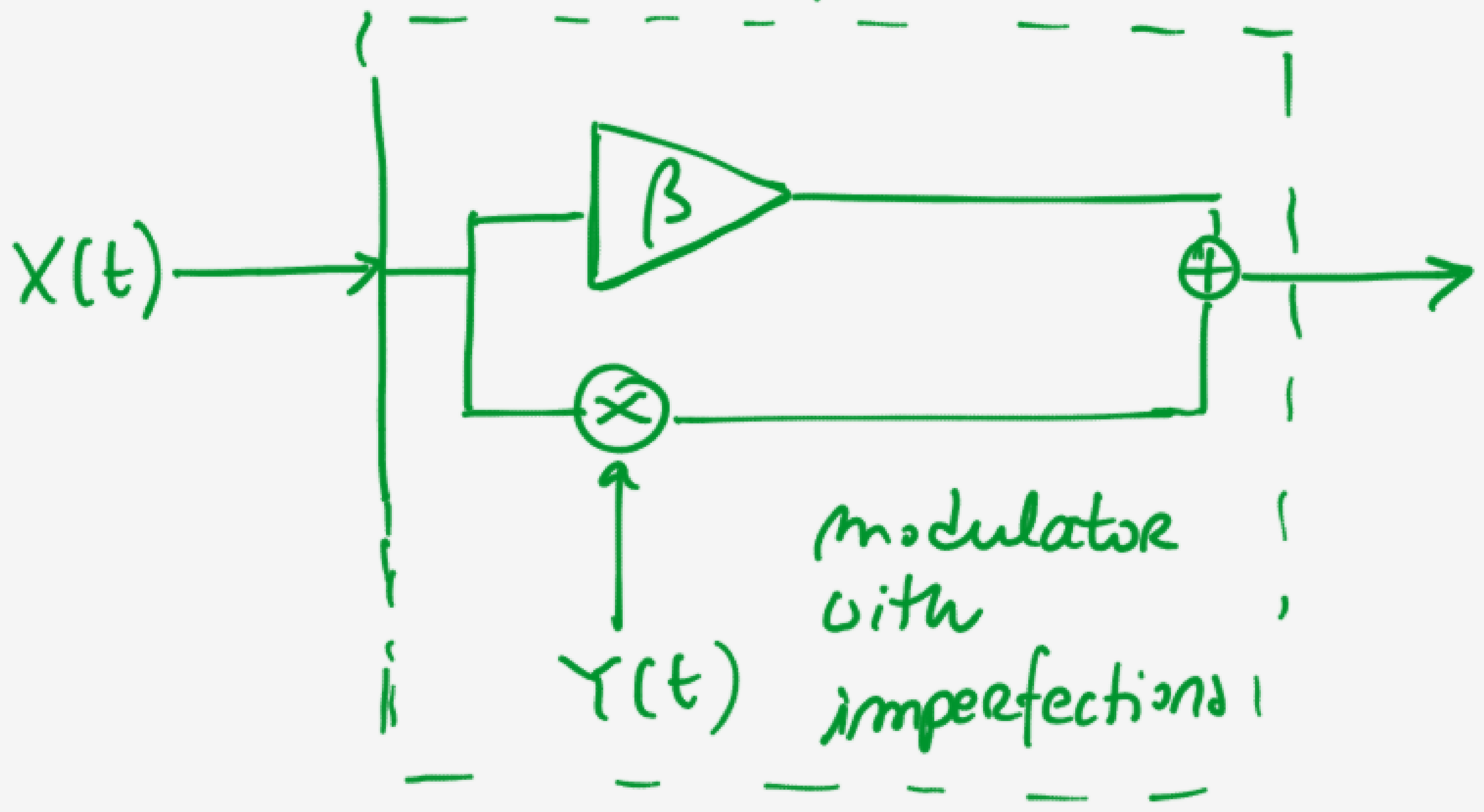


$Y(t) = \cos(2\pi f_0 t + \Phi)$

Φ uniformly distributed $[0, 2\pi)$

$X(t)$ is stationary, independent of $Y(t)$ and $m_X = 0$, $r_{XX}(\tau) = \sigma_X^2 e^{-\alpha|\tau|}$

With Hardware imperfections:



$\bar{Z}(t) = X(t)Y(t) + X(t)\beta$

(a) $ACF(\bar{Z}) = r_{\bar{Z}}(t+\tau, t) = E[(X(t+\tau)Y(t+\tau) + \beta Y(t+\tau)) \cdot (X(t)Y(t) + \beta Y(t))] = \{$ linearity + independence X and Y $\}$

$= E[X(t+\tau)X(t)]\,E[Y(t+\tau)Y(t)] + \beta E[\cancel{X(t)}]^{0} E[Y(t+\tau)Y(t)]$
$+ \beta \underbrace{E[\cancel{X(t+\tau)}]^{0}}_{\text{WSS and } \emptyset \text{mean}} E[Y(t+\tau)Y(t)] + \beta^2 E[Y(t+\tau)Y(t)] =$

$= R_{xx}(\tau) R_{yy}(\tau) + \beta^2 R_{yy}(\tau)$

· Obtain $R_{yy}(\tau)$:

$R_{yy}(\tau) = E[Y(t+\tau)Y(t)] = E[\cos(2\pi f_0(t+\tau) + \Phi)\cos(2\pi f_0 t + \Phi)] \rightarrow$

Φ is uniformly distributed within $[0, 2\pi)$, i.e.:

$$f_\Phi(\phi) = \frac{1}{2\pi} \qquad \phi \in [0, 2\pi)$$

· using that $\cos(a)\cos(b) = \dfrac{\cos(a+b) + \cos(a-b)}{2}$ we rewrite the expectation as:

$\frac{1}{2} E[\cos(2\pi f_0(2t+\tau) + 2\Phi) + \cos(2\pi f_0 \tau)] =$

$= \frac{1}{2}\cos(2\pi f_0 \tau) + \frac{1}{2} E[\cos(2\pi f_0(2t+\tau) + 2\Phi)] =$

$$= \frac{1}{2}\cos(2\pi f_0 \tau) + \frac{1}{2}\int_0^{2\pi} \cos(2\pi f_0(2t+\tau)+2\Phi)\cdot\frac{1}{2\pi}\,d\Phi =$$

$$= \frac{1}{2}\cos(2\pi f_0 \tau) + \frac{2}{\cancel{2}\cdot 2\pi}\sin(2\pi f_0(2t+\tau)+2\Phi)\Big]_0^{2\pi} =$$

$$= \frac{1}{2}\cos(2\pi f_0 \tau) + \frac{1}{2\pi}\underbrace{\left(\sin(2\pi f_0(2t+\tau)+4\pi) - \sin(2\pi f_0(2t+\tau))\right)}_{\emptyset} =$$

$$= \frac{1}{2}\cos(2\pi f_0 \tau).$$

Hence $R_{\tilde{Z}\tilde{Z}}(\tau) = R_{yy}(\tau)(R_{xx}(\tau)+\beta^2) = \frac{1}{2}\cos(2\pi f_0\tau)(\sigma_x^2 e^{-\alpha|\tau|}+\beta^2)$

(b) Spectrum can be found by just calculating the Fourier transform:

$R_{\tilde{Z}\tilde{Z}}(\tau) = \frac{1}{2}\cos(2\pi f_0\tau)(\sigma_x^2 e^{-\alpha|\tau|}+\beta^2)$

Fourier transforms (can be found in the pink booklet):

$$\cos(2\pi f_0\tau) \xrightarrow{\mathcal{F}} \frac{1}{2}(\delta(f+f_0)+\delta(f-f_0))$$

$$e^{-\alpha|\tau|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2+(2\pi f)^2} \qquad A \longrightarrow A\,\delta(f)$$

The product in time → convolution in frequency (and vice versa)

$$S_{\tilde{Z}\tilde{Z}}(f) = \mathcal{F}\{R_{ZZ}(\tau)\}(f) = \frac{1}{2}\mathcal{F}\{\cos(2\pi f_0\tau)\} * (\sigma_x^2\mathcal{F}\{e^{-\alpha|\tau|}\} + \mathcal{F}\{\beta^2\}) =$$

$$= \frac{1}{4}(\delta(f+f_0)+\delta(f-f_0)) * \left(\sigma_x^2\frac{2\alpha}{\alpha^2+(2\pi f)^2} + \beta^2\delta(f)\right) =$$

$$= \frac{1}{4}\sigma_x^2\left(\frac{2\alpha}{\alpha^2+(2\pi(f+f_0))^2} + \frac{2\alpha}{\alpha^2+(2\pi(f-f_0))^2}\right) + \frac{1}{4}\beta^2(\delta(f+f_0)+\delta(f-f_0))$$

(c) Calculate value of β such that $\frac{P_{\tilde{Z}}}{P_Z} = 2$

$P_{\tilde{Z}} = R_{\tilde{Z}\tilde{Z}}(0) = R_{yy}(0)(R_{xx}(0)+\beta^2) = \frac{1}{2}(\sigma_x^2+\beta^2)$

$P_Z = R_{\tilde{Z}\tilde{Z}}(0)\}$ with $\beta=\emptyset \Rightarrow R_{yy}(0)\cdot R_{xx}(0) = \frac{1}{2}\sigma_x^2$

$$\frac{P_{\tilde{Z}}}{P_Z} = \frac{\cancel{\frac{1}{2}}(\sigma_x^2+\beta^2)}{\cancel{\frac{1}{2}}\sigma_x^2} = \frac{\sigma_x^2+\beta^2}{\sigma_x^2} = 2 \qquad \sigma_x^2+\beta^2 = 2\sigma_x^2$$

$\Rightarrow \beta^2 = \sigma_x^2$